MOTION OF A GYROSCOPE WITH A FLEXIBLE AXIS ACTED ON BY GRAVITY AND ELASTIC CONSTRAINTS FOR SMALL ANGLES OF NUTATION. THE STABILITY OF ITS VERTICAL ROTATION

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The elastic deformations of the gyroscope rotor axis with high angular velocities can have a considerable effect on the motion of the gyroscope. One of the first studies of the dynamics of a gyroscope with an elastic axis was carried out by Magnus [1 and 2]. The oscillations of a free gyroscope with an elastic shaft were later investigated by Maunder and Whipple [3 and 4]. Krementulo [5] investigated the stability of motion of a gyroscope with allowance for the elastic properties of the rotor axis. These authors focused their attention on an astatic gyroscope in a universal suspension.

The present paper concerns the dynamics of a gyroscope with an elastic axis acted on by gravity and by the reactions of the elastic constraints. The gyroscope is considered as an extended symmetric body on a weightless elastic axis; some violation of symmetry by eccentrically situated point masses is considered acceptable. The quasilinear differential equations of motion of the gyroscope model are cited. The angular velocities of the forward and reverse precessions and the trajectories of the center of mass of the symmetric gyroscope are determined for small angles of nutation. The forced oscillations of the gyroscope due to an eccentric point mass and its critical velocities are investigated. The dynamic characteristics of gyroscopes with elastic and absolutely rigid axes are compared.

If the center of mass of the gyroscope lies above the point of support, there arises the problem of stability of the vertical rotation of the gyroscope with elastic axis in the presence of elastic constraints. The necessary conditions for stability of the vertical rotation in this case are developed. The stability of an elastic top is investigated as an example. It is shown that the elastic deformation of the axis raises the threshold of the rotor angular velocity below which the rotation becomes unstable. The dependence of this threshold on the moments of inertia of the top and on the elastic properties of its axis is cited.

The model of a gyroscope adopted in the present study was chosen to make possible the solution of several practical problems.

1. We represent our gyroscope in the form of a heavy symmetric absolutely solid body mounted on a flexible shaft of negligibly small mass (Fig. 1). The point of support O of the gyroscope is fixed. The mass of the body is m_1, its polar moment of inertia is A_1 and the equatorial moments of inertia with respect to the central axes are A_2. The

distance OO_1 between the center of inertia of the mass m_1 and the point of support is l ; the length of the elastic shaft is l_1.

Neglecting the variation of the quantities l and l_1 with deformation of the axis, we specify the position of the center of inertia of the mass m_1 relative to the fixed axes ξ_1, ξ_2, ξ_3 by means of two spherical coordinates, i.e. by the angles γ and θ. The unit vectors of the trihedron $O_1x_1x_2x_3$ of the spherical axes are denoted by $\mathbf{k}_1$, $\mathbf{k}_2$ and $\mathbf{k}_3$; the unit vectors of the fixed axes ξ_1, ξ_2, ξ_3 are denoted by $\mathbf{j}_1$, $\mathbf{j}_2$ and $\mathbf{j}_3$. The projections of the transverse deflections $u_1(s, t)$ and $u_2(s, t)$ of the elastic line onto the coordinate planes x_1x_3 and x_2x_3 are measured from the straight line OO_1 and are considered positive if their directions are the same as those of the unit vectors $\mathbf{k}_1$ and $\mathbf{k}_2$. The elastic constraint near the point of support O produces a restoring moment proportional to the angle between the vertical and the tangent to the elastic rotor axis at this point; the moment vector is perpendicular to the plane formed by the two indicated straight lines.

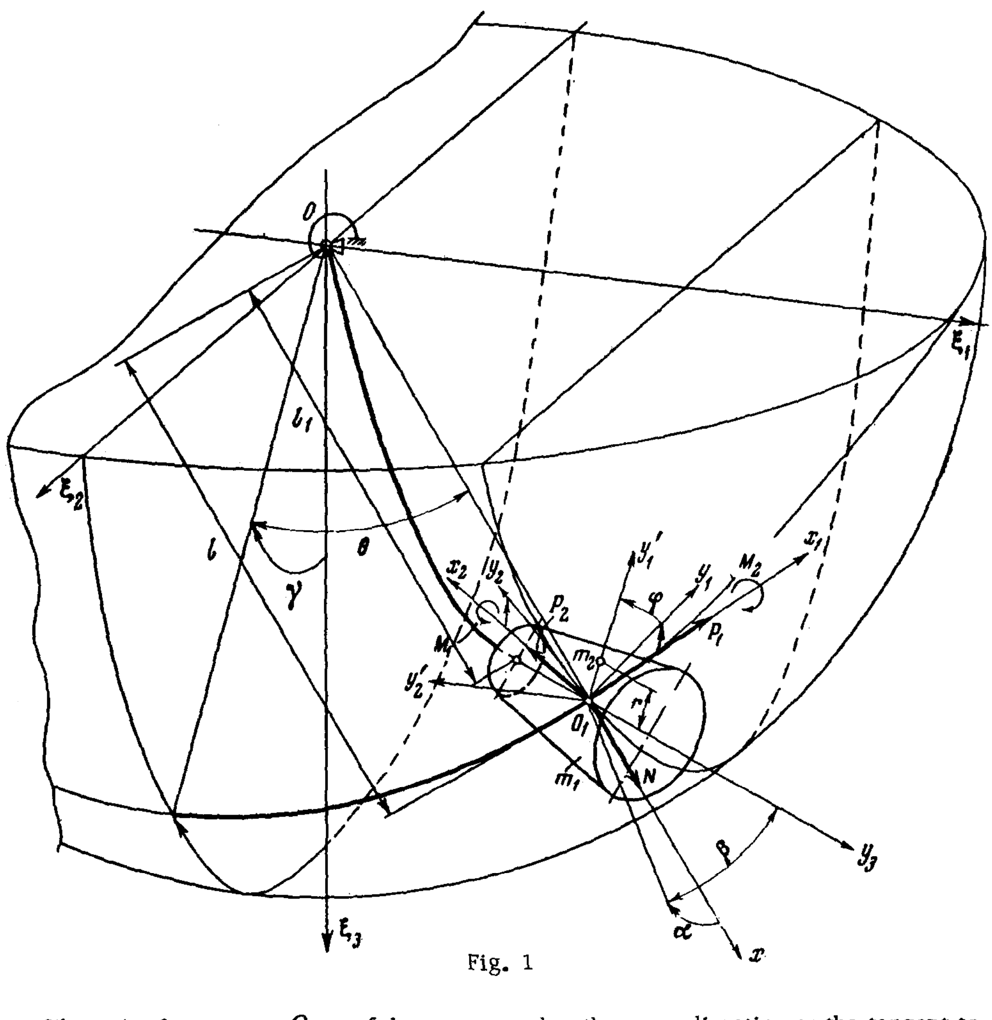

Fig. 1

The axis of symmetry O_1y_3 of the gyroscope has the same direction as the tangent to the elastic axis at the point $s = l_1$; its position relative to the spherical coordinate system is defined by the Résal angles α and β; the same angles define the positions of the Résal axes y_1, y_2, y_3 whose unit vectors we denote by $\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$. Finally, the position of the trihedron of axes $O_1y_1'y_2'y_3$ relative to the trihedron of Résal axes $O_1y_1y_2y_3$ is defined by the angle of proper rotation φ.

We assume that the complete symmetry of the gyroscope can be violated by eccentrically placed point masses; their effect on the dynamics of the gyroscope can be allowed for by adding the gravitational forces and the inertial forces acting on the point masses in their absolute motion to the loads acting on the symmetric gyroscope.

To simplify our formulas we shall henceforth consider just one point mass m_2 $(m_2 \ll m_1)$ with the coordinates $y_1' = r$, $y_2' = y_3 = 0$.

Bearing in mind our intention to consider the motion of the gyroscope for small angles of nutation, in all of the nonlinear functions we retain terms of up to the third order of smallness in the coordinates γ, θ and their derivatives, but only first-order terms in the quantities characterizing the elastic deformations of the axis.

The acceleration $\mathbf{w}_1$ of the center of inertia of the mass m_1 within the above degree of accuracy is given by

$$\mathbf{w}_1 = l(\theta^{\cdot\cdot} + \theta\gamma^{\cdot 2})\mathbf{k}_1 + l(\gamma^{\cdot\cdot} - 2\theta\gamma^{\cdot}\theta^{\cdot} - {}^1/_2\gamma^{\cdot\cdot}\theta^2)\mathbf{k}_2 - l(\gamma^{\cdot 2} + \theta^{\cdot 2})\mathbf{k}_3$$

The angular velocity of the body is

$$\begin{aligned}\boldsymbol{\Omega} = -\gamma^{\cdot}\mathbf{j}_1 - \alpha^{\cdot}\mathbf{k}_1 + \theta^{\cdot}\mathbf{k}_2 + \beta^{\cdot}\mathbf{i}_2 + \varphi^{\cdot}\mathbf{i}_3 = (-\gamma^{\cdot}\cos\theta - \alpha^{\cdot} + \varphi^{\cdot}\sin\beta)\mathbf{k}_1 + \\ + (\theta^{\cdot} + \beta^{\cdot}\cos\alpha + \varphi^{\cdot}\sin\alpha\cos\beta)\mathbf{k}_2 + (-\gamma^{\cdot}\sin\theta - \beta^{\cdot}\sin\alpha + \\ + \varphi^{\cdot}\cos\alpha\cos\beta)\mathbf{k}_3 \approx (-\alpha^{\cdot} - \gamma^{\cdot} + \beta\varphi^{\cdot} + {}^1/_2\theta^2\gamma^{\cdot})\mathbf{k}_1 + \\ + (\theta^{\cdot} + \beta^{\cdot} + \alpha\varphi^{\cdot})\mathbf{k}_2 + (\varphi^{\cdot} - \theta\gamma^{\cdot})\mathbf{k}_3\end{aligned}$$

The acceleration of the point mass m_2 is

$$\mathbf{w}_2 = \mathbf{w}_1 + d/dt\,[\boldsymbol{\Omega} \times \mathbf{r}], \qquad \mathbf{r} = r\cos\varphi\mathbf{i}_1 + r\sin\varphi\mathbf{i}_2$$

The force $\mathbf{P}$ applied to the elastic axis of the gyroscope at the point O_1 is

$$\mathbf{P} = mg\mathbf{j}_3 - m\mathbf{w}_1 - d/dt\,[\boldsymbol{\Omega} \times m_2\mathbf{r}], \qquad m = m_1 + m_2$$

Resolving the vector $\mathbf{P}$ along the axes x_1, x_2, x_3 and isolating the first-order terms, we find that within the indicated degree of accuracy

$$\mathbf{P} = P_1\mathbf{k}_1 + P_2\mathbf{k}_2 + N\mathbf{k}_3 \tag{1.1}$$

$$P_1 = P_1^\circ + f_1, \qquad P_2 = P_2^\circ + f_2, \qquad N = mg(1 + f_3)$$

$$P_1^\circ = -mg\theta - ml\theta^{\cdot\cdot} + \varepsilon\varphi^{\cdot 2}\cos\varphi + \varepsilon\varphi^{\cdot\cdot}\sin\varphi$$

$$P_2^\circ = -mg\gamma - ml\gamma^{\cdot\cdot} + \varepsilon\varphi^{\cdot 2}\sin\varphi - \varepsilon\varphi^{\cdot\cdot}\cos\varphi$$

$$f_1 = mg({}^1/_2\theta\gamma^2 + {}^1/_6\theta^3) - ml\theta\gamma^{\cdot 2}, \qquad f_2 = mg(\gamma^3/6) + m({}^1/_2\,l\theta^2\gamma^{\cdot\cdot} + 2l\theta\gamma^{\cdot}\theta^{\cdot})$$

$$\begin{aligned}f_3 = l/g\,(\gamma^{\cdot 2} + \theta^{\cdot 2}) - {}^1/_2(\gamma^2 + \theta^2) + (\varepsilon/mg)\sin\varphi\,(\alpha^{\cdot\cdot} + \gamma^{\cdot\cdot} - 2\beta^{\cdot}\varphi^{\cdot} - 2\theta^{\cdot}\varphi^{\cdot} - \\ - \alpha\varphi^{\cdot 2} - \beta\varphi^{\cdot\cdot}) + (\varepsilon/mg)\cos\varphi\,(\beta^{\cdot\cdot} + \theta^{\cdot\cdot} + 2\alpha^{\cdot}\varphi^{\cdot} + 2\gamma^{\cdot}\varphi^{\cdot} - \beta\varphi^{\cdot 2} + \alpha\varphi^{\cdot\cdot})\end{aligned}$$

where the static moment of the eccentric mass $\varepsilon = m_2r$ is assumed to be small.

The projections L_1 and L_2 on the principal axes of inertia y_1 and y_2 of the moment vector $\mathbf{L}$ applied to the shaft at the point O_1 are given by

$$L_1 = -A_2\frac{d\Omega_1}{dt} - A_1\omega_{12}\Omega_3 + A_2\Omega_2\omega_{13} + m_2\mathbf{r}\times(g\mathbf{j}_3 - \mathbf{w}_2)\mathbf{i}_1 \qquad (1.2)$$

$$L_2 = -A_2\frac{d\Omega_2}{dt} + A_1\omega_{11}\Omega_3 - A_2\Omega_1\omega_{13} + m_2\mathbf{r}\times(g\mathbf{j}_3 - \mathbf{w}_2)\mathbf{i}_2$$

where $\Omega_1, \Omega_2, \Omega_3$ and $\omega_{11}, \omega_{12}, \omega_{13}$ are the projections on the axes y_1, y_2, y_3 of the angular velocities of the body m_1 and of the trihedron of axes $O_1y_1y_2y_3$, resectively, i. e.

$$\Omega_1 = \omega_{11} = -\alpha^\cdot \cos\beta - \theta^\cdot \sin\alpha\sin\beta - \gamma^\cdot\cos\theta\cos\beta + \qquad (1.3)$$

$$+\gamma^\cdot\sin\theta\cos\alpha\sin\beta \approx -\alpha^\cdot - \gamma^\cdot + {}^1/_2\gamma^\cdot\theta^2$$

$$\Omega_2 = \omega_{12} = \beta^\cdot + \theta^\cdot\cos\alpha + \gamma^\cdot\sin\alpha\sin\theta \approx \beta^\cdot + \theta^\cdot$$

$$\omega_{13} = \theta^\cdot\sin\alpha\cos\beta - \alpha^\cdot\sin\beta - \gamma^\cdot\cos\theta\sin\beta - \gamma^\cdot\sin\theta\cos\alpha\cos\beta \approx -\gamma^\cdot\theta$$

$$\Omega_3 = \omega_{13} + \varphi^\cdot$$

Projecting the moment vector $\mathbf{L}$ onto the axes x_1 and x_2, we find that at the point O_1 the gyroscope shaft is acted on by the moments M_1 and M_2 in the planes x_1x_3 and x_2x_3. These moments are given by

$$M_1 = L_2\cos\alpha - L_1\sin\alpha\sin\beta \approx L_2, \qquad M_2 = -L_1\cos\beta \approx -L_1$$

The moments M_1 and M_2 are considered positive if they bend the axis in the same direction as do the positive forces P_1 and P_2. Substituting (1. 3) into (1. 2) and isolating the first-order terms, we obtain

$$M_1 = M_1^\circ + f_4, \qquad M_2 = M_2^\circ + f_5 \qquad (1.4)$$

$$M_1^\circ = -A_2(\beta^{\cdot\cdot} + \theta^{\cdot\cdot}) - A_1\varphi^\cdot(\alpha^\cdot + \gamma^\cdot) - \varepsilon g\cos\varphi$$

$$f_4 = {}^1/_2A_1\varphi^\cdot\gamma^\cdot\theta^2 + \theta\gamma^{\cdot 2}(A_1 - A_2)$$

$$M_2^\circ = -A_2(\alpha^{\cdot\cdot} + \gamma^{\cdot\cdot}) + A_1\varphi^\cdot(\beta^\cdot + \theta^\cdot) - \varepsilon g\sin\varphi$$

$$f_5 = {}^1/_2A_2\theta^2\gamma^{\cdot\cdot} + (2A_2 - A_1)\theta\gamma^\cdot\theta^\cdot$$

From the equation of moments with respect to the axis y_3 (neglecting friction at the support), we obtain

$$A_1\frac{d\Omega_3}{dt} = m_2\mathbf{r}\times(g\mathbf{j}_3 - \mathbf{w}_2)\mathbf{i}_3 \qquad (1.5)$$

or

$$(A_1 + m_2r^2)\varphi^{\cdot\cdot} \approx A_2(\gamma^\cdot\theta)^\cdot + \varepsilon(g\theta + l\theta^{\cdot\cdot})\sin\varphi - \varepsilon(g\gamma + l\gamma^{\cdot\cdot})\cos\varphi$$

The remaining four equations of motion can be obtained from the following self-evident equations valid for a weightless shaft:

$$P_1l + M_1 - \varkappa[\theta + u_1'(0,t)] = 0, \qquad \beta = u_1'(l_1, t)$$

$$P_2l + M_2 - \varkappa[\gamma + u_2'(0,t)] = 0, \qquad \alpha = u_2'(l_1, t) \qquad (1.6)$$

where $\varkappa$ is the rigidity of the elastic constraint and where the primes next to the $u_j(s, t)$ $(j = 1, 2)$ denote partial derivatives with respect to s.

The projections of the deflections $u_j(s, t)$ of the gyroscope shaft axis on the coordinates x_1x_3 and x_2x_3 must satisfy the differential Eqs.

$$EIu_j''(s,t) - Nu_j(s,t) = P_j(l - s) + M_j \quad \text{for} \quad 0 < s < l_1 \quad (j = 1, 2) \qquad (1.7)$$

the relations

$$u_j(s,t) = u_j(l_1,t) + (s-l_1)u_j'(l_1,t) \quad \text{for } l_1 < s < l$$

and the boundary conditions

$$u_j(0,t) = 0, \quad u_j(l,t) = u_j(l_1,t) + (l-l_1)u_j'(l_1,t) = 0 \tag{1.8}$$

We introduce the function λ given by

$$\lambda = \left(\frac{N}{EI}\right)^{1/2} = \lambda_0\sqrt{1+f_3}, \quad \lambda_0 = \left(\frac{mg}{EI}\right)^{1/2} \tag{1.9}$$

where EI is the constant bending rigidity of the shaft. Integrating Eq.(1.7) under boundary conditions (1.8), we obtain the following expressions for the angles of inclination of the tangent to the elastic axis for $s = 0$ and $s = l_1$:

$$u_j'(0,t) = N^{-1}[P_jc_1(\lambda) + l^{-1}M_jc_2(\lambda)]$$
$$u_j'(l,t) = u_j'(l_1,t) = N^{-1}[P_jc_3(\lambda) + l^{-1}M_jc_4(\lambda)] \tag{1.10}$$

where the dimensionless coefficients $c_k(\lambda)$ $(k = 1, 2, 3, 4)$ are:

$$c_1(\lambda) = 1 - \frac{\lambda l}{c(\lambda)}[\operatorname{ch}\lambda l_1 + \lambda(l-l_1)\operatorname{sh}\lambda l_1], \quad c_3(\lambda) = 1 - \frac{\lambda l}{c(\lambda)}$$
$$c_2(\lambda) = \frac{\lambda l}{c(\lambda)}[1 - \operatorname{ch}\lambda l_1 - \lambda(l-l_1)\operatorname{sh}\lambda l_1], \quad c_4(\lambda) = \frac{\lambda l}{c(\lambda)}(\operatorname{ch}\lambda l_1 - 1)$$
$$c(\lambda) = \operatorname{sh}\lambda l_1 + \lambda(l-l_1)\operatorname{ch}\lambda l_1 \tag{1.11}$$

Let us expand the functions $u_j'(0,t)$ and $u_j'(l,t)$ in Taylor series in the parameter f_3. Since f_3 is a second-order quantity in α, β, γ, θ and ε, we retain only the linear terms in f_3 in these expansions,

$$mgu_j'(0,t) = [c_1(\lambda_0) + f_3d_1(\lambda_0)]P_j + l^{-1}M_j[c_2(\lambda_0) + f_3d_2(\lambda_0)]$$
$$mg\,u_j'(l,t) = [c_3(\lambda_0) + f_3d_3(\lambda_0)]P_j + l^{-1}M_j[c_4(\lambda_0) + f_3d_4(\lambda_0)]$$

where, as we can readily verify,

$$d_k = \frac{\lambda_0}{2}\frac{dc_k(\lambda_0)}{d\lambda_0} - c_k(\lambda_0), \quad \lambda_0 = \left(\frac{mg}{EI}\right)^{1/2} \qquad (k = 1, 2, 3, 4)$$

Substituting the values of the angles of inclination of $u_j'(0,t)$ and $u_j'(l,t)$ into Eqs.(1.6), we obtain

$$P_1^\circ l(1-\eta c_1) + M_1^\circ(1-\eta c_2) - \varkappa\theta = F_1$$
$$P_2^\circ l(1-\eta c_1) + M_2^\circ(1-\eta c_2) - \varkappa\gamma = F_2 \tag{1.12}$$
$$mg\beta - c_3P_1^\circ - l^{-1}c_4M_1^\circ = F_3 \qquad \left(\eta = \frac{\varkappa}{mgl}\right)$$
$$mg\alpha - c_3P_2^\circ - l^{-1}c_4M_2^\circ = F_4$$

Here P_j° and M_j° are linear functions of the coordinates given by Formulas (1.1) and (1.4); F_k are functions not containing first-order terms and given by

$$F_1 = f_1l(\eta c_1 - 1) + f_3\eta(ld_1P_1^\circ + d_2M_1^\circ) + f_4(\eta c_2 - 1)$$
$$F_2 = f_2l(\eta c_1 - 1) + f_3\eta(l\,d_1P_2^\circ + d_2M_2^\circ) + f_5((\eta c_2 - 1)$$
$$F_3 = f_1c_3 + f_3(d_3P_1^\circ + l^{-1}d_4M_1^\circ) + l^{-1}f_4c_4$$
$$F_4 = f_2c_3 + f_3(d_3P_2^\circ + l^{-1}d_4M_1^\circ) + l^{-1}f_5c_4$$

Quasilinear Eqs. (1.5) and (1.2) describe completely the motion of a gyroscope with an elastic axis for small angles of nutation. We shall confine ourselves to an investigation of the linearized equations.

Neglecting second-order terms in Eqs. (1.5), we denote the constant angular velocity of the proper rotation by ω.

Introducing the complex functions

$$x = \beta + i\alpha, \qquad y = \theta + i\gamma \tag{1.13}$$

and the dimensionless parameters

$$\sigma^2 = \frac{A_2}{ml^2}, \qquad \sigma_0^2 = \frac{A_1}{ml^2} \tag{1.14}$$

we can express linearized equations of motion (1.12) as

$$c_4\sigma^2 x^{\cdot\cdot} + (c_3 + c_4\sigma^2)\, y^{\cdot\cdot} + \frac{g}{l} x + \frac{g}{l} c_3 y - c_4\sigma_0^2\omega i\,(x^{\cdot} + y^{\cdot}) = \frac{\varepsilon\omega^2}{ml}\left(c_3 - \frac{gc_4}{\omega^2 l}\right) e^{i\omega t}$$

$$(1 - \eta c_2)\,\sigma^2 x^{\cdot\cdot} + [1 + \sigma^2 - \eta\,(c_1 + c_2\sigma^2)]\, y^{\cdot\cdot} + \frac{g}{l}\,[1 + \eta\,(1 - c_1)]\, y -$$
$$- (1 - \eta c_2)\,\sigma_0^2\omega i\,(x^{\cdot} + y^{\cdot}) = \frac{\varepsilon\omega^2}{ml}\left(1 - \eta c_1 - \frac{g}{l}\,\frac{1 - \eta c_2}{\omega^2}\right) e^{i\omega t} \tag{1.15}$$

Attempting to find a solution of homogeneous system (1.15) of the form

$$x = D_1 e^{i\nu t}, \qquad y = D_2\, e^{i\nu t} \tag{1.16}$$

we obtain the frequency equation

$$(\nu^4\sigma^2 - \nu^3\omega\sigma_0^2)\,[c_4 - c_3 + \eta\,(c_2c_3 - c_1c_4)] - gl^{-1}\,\nu^2\,\{1 + \sigma^2\,(1 - c_3 +$$
$$+ c_4) - \eta\,[c_1 + \sigma^2\,(c_2 - c_4 + c_1c_4 - c_2c_3)]\} + gl^{-1}\sigma_0^2\omega\nu\,[1 - c_3 + c_4 + \eta\,(c_4 - c_2 +$$
$$+ c_2c_3 - c_1c_4)] + g^2l^{-2}\,[1 + \eta\,(1 - c_1)] = 0$$

We introduce the abstract quantities

$$\nu_* = \nu\sqrt{l/g}, \qquad \omega_* = \omega\sqrt{l/g} \tag{1.17}$$

and replace the coefficients c_k by their values from (1.11). By elementary transformations we reduce the frequency equation to the form

$$a_0\nu_*^4 + a_1\nu_*^3 + a_2\nu_*^2 + a_3\nu_* + a_4 = 0 \tag{1.18}$$

where

$$a_0 = \sigma^2 c^{-1}\,[\vartheta\,\mathrm{ch}\,\vartheta_1 - c + \eta\vartheta\,(2 - 2\,\mathrm{ch}\vartheta_1 + \vartheta_1\mathrm{sh}\vartheta_1],\quad a_1 = -\,\omega_* a_0\sigma_0^2/\sigma^2$$
$$a_2 = -\,(1 + c^{-1}\,\sigma^2\vartheta\mathrm{ch}\vartheta_1) + \eta\,\{1 - \vartheta c^{-1}\,[\mathrm{ch}\vartheta_1 + (\vartheta - \vartheta_1)\,\mathrm{sh}\vartheta_1] -$$
$$- c^{-1}\sigma^2\vartheta^2\mathrm{sh}\vartheta_1\} \tag{1.19}$$

$$a_3 = c^{-1}\,\sigma_0^2\vartheta\omega_*(\mathrm{ch}\,\vartheta_1 + \eta\vartheta\,\mathrm{sh}\,\vartheta_1),\qquad a_4 = 1 + \eta\vartheta c^{-1}[\,\mathrm{ch}\vartheta_1 + (\vartheta - \vartheta_1)\,\mathrm{sh}\,\vartheta_1]$$
$$\vartheta = \lambda_0 l,\qquad \vartheta_1 = \lambda_0 l_1,\qquad c = \mathrm{sh}\,\vartheta_1 + (\vartheta - \vartheta_1)\,\mathrm{ch}\,\vartheta_1$$

Formulas (1.19) for the coefficients of the frequency equation are simplified if (as is often the case) $l/l_1 \approx 1$, and if

$$a_0 = \sigma^2\,[\vartheta\mathrm{cth}\vartheta - 1 + \eta\vartheta\,(\vartheta - 2\mathrm{th}\,\vartheta/2)],\quad a_1 = -\omega_*\sigma_0^2 a_0/\sigma^2$$
$$a_2 = -(1 + \sigma^2\,\vartheta\,\mathrm{cth}\,\vartheta) + \eta\,(1 - \vartheta\mathrm{cth}\,\vartheta - \sigma^2\vartheta^2)$$
$$a_3 = \omega_*\sigma_0^2\vartheta\,(\mathrm{cth}\,\vartheta + \eta\vartheta),\qquad a_4 = 1 + \eta\vartheta\mathrm{cth}\,\vartheta \tag{1.20}$$

The general solution of homogeneous system (1.15) is

$$y = \sum_{k=1}^{4} R_k e^{i(\nu_k t+\psi_k)}, \qquad x = \sum_{k=1}^{4} R_k q_k e^{i(\nu_k t+\psi_k)}$$

Here ν_k are the natural frequencies of the system, $\nu_k = \nu_{*k}\sqrt{g/l}$ [ν_{*k} are the roots of frequency Eq. (1.18)] and q_k are the coefficients of the vibrational modes,

$$q_k = \frac{(c_3 + c_4\sigma^2)\nu_k^2 - c_4\sigma_0^2\omega\nu_k - c_3 g/l}{-c_4\sigma^2\nu_k^2 + c_4\sigma_0^2\omega\nu_k + g/l}$$

R_k and ψ_k are real arbitrary constants which can be expressed in terms of the initial angles α_0, β_0, γ_0, θ_0 and velocities $\alpha_0^{\cdot}$, $\beta_0^{\cdot}$, $\gamma_0^{\cdot}$, $\theta_0^{\cdot}$ according to Formulas

$$R_k = \sqrt{\rho_k^2 + s_k^2}, \quad \sin\psi_k = \frac{s_k}{R_k}, \quad \cos\varphi_k = \frac{\rho_k}{R_k}$$

where

$$\begin{aligned} \Delta\rho_k (-1)^{k+1} &= \theta_0\Delta_{1k} - \gamma_0^{\cdot}\Delta_{2k} + \beta_0\Delta_{3k} - \alpha_0^{\cdot}\Delta_{4k} \\ \Delta s_k (-1)^{k+1} &= \gamma_0\Delta_{1k} + \theta_0^{\cdot}\Delta_{2k} + \alpha_0\Delta_{3k} + \beta_0^{\cdot}\Delta_{4k} \end{aligned} \tag{1.21}$$

In Formulas (1.21)

$$\Delta = \begin{vmatrix} 1 & 1 & 1 & 1 \\ \nu_1 & \nu_2 & \nu_3 & \nu_4 \\ q_1 & q_2 & q_3 & q_4 \\ q_1\nu_1 & q_2\nu_2 & q_3\nu_3 & q_4\nu_4 \end{vmatrix}$$

and Δ_{jk} are the minors of the elements of the determinant Δ, situated at the intersection of the jth row and the kth column.

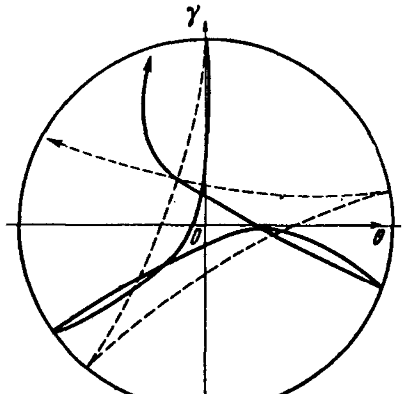

Fig. 2

Thus, the projection of the trajectory of the center of inertia of the symmetric gyroscope with elastic axis onto the horizontal plane for small angles of nutation is the geometric locus of the vector equal to the sum of four vectors, each of which rotates with the angular velocity ν_k ($k = 1, 2, 3, 4$), to describe a circle of radius R_k.

The effect of rotor axis deformation on the trajectory of the center of inertia of a gyropendulum can be analyzed by means of specific examples. Fig. 2 shows a loop of the trajectory of the center of inertia of a gyropendulum without an elastic constraint with a flexible shaft for $\vartheta = \vartheta_1 = 1.5$, $\sigma = 0.75$, $\sigma_0 = 0.75\sqrt{2}$, $\omega = 0.5$ (in fraction of $\sqrt{g/l}$) under the initial conditions $\gamma = \gamma_0$, $\alpha_0 = \beta_0 = \theta_0 = \alpha_0^{\cdot} = \beta_0^{\cdot} = \gamma_0^{\cdot} = \theta_0^{\cdot} = 0$, for which the angular velocities of precession from (1.18) are $\nu_1 = -1.96$, $\nu_2 = -0.55$, $\nu_3 = 1.00$, $\nu_4 = 2.50$. A loop of the trajectory of the center of inertia of the same gyropendulum under the same initial conditions but with an absolutely rigid shaft appears as the broken curve in Fig. 2.

The steadystate oscillations of the gyroscope due to the point mass m_2 can be obtained as particular solutions of system (1.15). Omitting terms containing ω^2 in the denominator from the right sides of these equations, we obtain the perticular solutions

$$X = \frac{\varepsilon\omega^2}{ml\Delta_1}\left[-a_0\omega^2\left(\frac{\sigma_0^2}{\sigma^2}-1\right)+\eta\frac{g}{l}c_3\right]e^{i\omega t} \qquad (1.22)$$

$$Y = \frac{\varepsilon\omega^2}{ml\Delta_1}\left[a_0\omega^2\left(\frac{\sigma_0^2}{\sigma^2}-1\right)+\frac{g}{l}(1-\eta c_1)\right]e^{i\omega t}$$

$$\Delta_1 = a_0\omega^4\left(1-\frac{\sigma_0^2}{\sigma^2}\right)+a_5\frac{g}{l}\omega^2+a_4\frac{g^2}{l^2}$$

The coefficients c_1 and c_3 in (1.22) can be determined from (1.11); the coefficients a_0, a_4 can be determined from (1.19); the coefficient a_5 is given by

$$a_5 = -1+\frac{\vartheta\,\mathrm{ch}\,\vartheta_1}{c}(\sigma_0^2-\sigma^2)+$$

$$+\frac{\eta}{c}\left[(1-\vartheta^2+\vartheta\vartheta_1)\,\mathrm{sh}\,\vartheta_1-\vartheta_1\,\mathrm{ch}\,\vartheta_1+\vartheta^2(\sigma_0^2-\sigma^2)\,\mathrm{sh}\,\vartheta_1\right]$$

By analogy with elastic rotors we can speak of the critical velocities of gyrosystems at which the polynomial $\Delta_1(\omega)$ is equal to zero. Let us consider in more detail the case where there is no elastic constraint in the system and where $l \approx l_1$; in this case we have

$$\Delta_1(\omega)=(\vartheta\,\mathrm{cth}\,\vartheta-1)(\sigma^2-\sigma_0^2)\,\omega^4+[\vartheta(\sigma_0^2-\sigma^2)\,\mathrm{cth}\,\vartheta-1]\,gl^{-1}\omega^2+g^2l^{-2}$$

The biquadratic equation $\Delta_1(\omega)=0$ does not have real roots if the coefficients of both ω^4 and ω^2 are positive or if the discriminant

$$4g^2l^{-2}(\sigma_0^2-\sigma^2)-g^2l^{-2}[1+\vartheta(\sigma_0^2-\sigma^2)\,\mathrm{cth}\,\vartheta]^2>0$$

In the case of an elastic axis the parameter $\vartheta\,\mathrm{cth}\,\vartheta$ is greater than unity, and none of these conditions is fulfilled. Hence, a gyropendulum with an elastic axis cannot be free of critical velocities; there are two such velocities for $\sigma^2 > \sigma_0^2$ and one for $\sigma^2 < \sigma_0^2$.

Let us compare the angular velocities of precession of the gyroscope with and without allowance for the elasticity of its axis. For simplicity we consider the massive part of the gyroscope rotor as a flywheel with the ratio of moments of inertia $A_1/A_2 = 2$ moreover, we neglect the thickness of the flywheel as compared with the length of the elastic weightless axis. From (1.18) and (1.19) we find that with allowance for axis elasticity the angular velocities of precession ν can be determined from Eq.

$$\sigma^2[\vartheta\,\mathrm{cth}\,\vartheta-1+\eta\vartheta(\vartheta-2\,\mathrm{th}\,{}^1\!/_2\vartheta)](\nu^4-2\omega\nu^3)-[1+\sigma^2\vartheta\,\mathrm{cth}\,\vartheta+$$

$$+\eta(\vartheta\,\mathrm{cth}\,\vartheta-1+\sigma^2\vartheta^2]\,\nu^2 g/l+2\sigma^2\omega(\vartheta\,\mathrm{cth}\,\vartheta+\eta\vartheta^2)\,\nu\, g/l+$$

$$+(1+\eta\vartheta\,\mathrm{cth}\,\vartheta)\,g^2/l^2=0$$

For a gyroscope with a nondeformable axis $\vartheta=0$, $\vartheta\,\mathrm{cth}\,\vartheta=$, and its angular velocities of precession ν_0 must satisfy Eq.

$$(1+\sigma^2)\,\nu_0^2-2\omega\sigma^2\nu_0-(1+\eta)\,g/l=0$$

The angular velocity ratios ν/ν_0 depend on the dimensionless parameters ϑ, σ, η and on the angular velocity ω of the rotor. These relations can be represented as a two-parameter family of surfaces. Fig. 3 shows such a surface for the ratio of the minimal angular velocities of forward precession of the gyroscope for $\eta=5$ and $\omega=0.5$ [in fractions of $(g/l)^{1/2}$]. As we see from the figure, the effect of axis bending on the precessional velocity for these numerical values of η and ω is especially significant in the ranges $0<\vartheta<3$ and $0<\sigma<1$.

It is no less interesting to compare the precessional velocity ν_k of the gyroscope model under consideration with the natural frequencies p_k of the bending oscillations of a similar model of a horizontal rotor. It is easy to show that the frequencies p_k are given by Eq.

$$\sigma^2\vartheta^2(\eta\vartheta^2+4)(p^4-2p^3\omega)-4p^2gl^{-1}[(\eta\vartheta^2+1)(3\sigma^2+1)+2]+$$
$$+24gl^{-1}\sigma^2\omega p(\eta\vartheta^2+1)+12\eta g^2l^{-2}=0$$

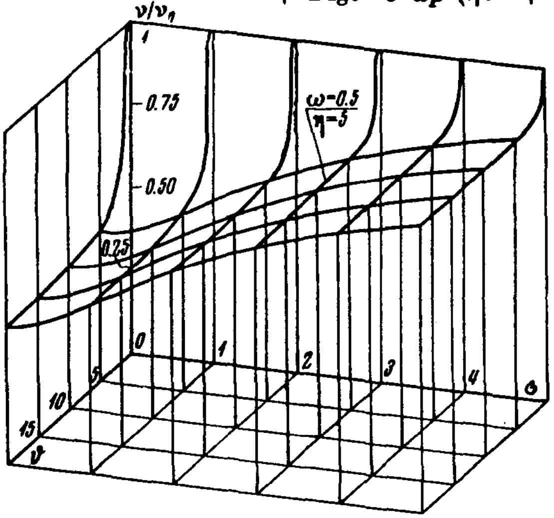

Fig. 3

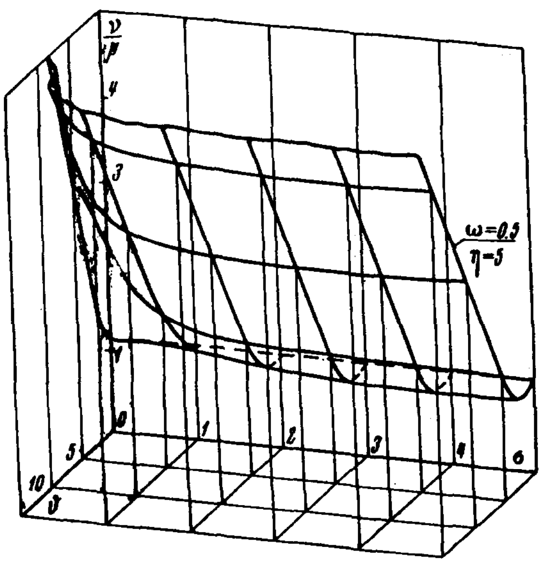

Fig. 4

Fig. 4 shows the ratios ν/p of the minimal velocities of forward precession of the gyroscope and the natural frequency of the horizontal rotor as functions of ϑ and σ for the same parameter values $\eta=5$ and $\omega=0.5\,(g/l)^{1/2}$. For small ϑ close to zero, this ratio differs little from unity; it then increases rapidly with increasing ϑ, largely due to the tensile longitudinal forces which increase the bending rigidity of the gyroscope shaft. This property of vertical rotors is exploited in high-speed ultracentrifuges whose rotors are made thin and flexible. Even with attached components of relatively small mass the parameter ϑ assumes large values, and the natural frequencies p_j of the bending vibrations can sometimes be made to exceed the operating rpm's.

2. Placing the center of masses above the point of support, gives rise to the problem of stability of vertical rotation of a gyroscope with an elastic axis. We assume that the gyroscope is completely symmetric ($m_2=0$).

To determine the necessary conditions for stability of the vertical position of the axis of symmetry of the gyroscope, we derive the equations in variations for the steadystate solution

$$\alpha=\beta=\gamma=\theta=\alpha^{\cdot}=\beta^{\cdot}=\gamma^{\cdot}=\theta^{\cdot}=0 \tag{2.1}$$

We direct the axis $O\xi_3$ vertically upwards (Fig. 5). To within first-order quantities the projections of the spherical axes of the force applied to the rotor at the point O_1 are

$$P_1=mg\theta-ml\theta^{\cdot\cdot},\quad P_2=mg\gamma-ml\gamma^{\cdot\cdot},\quad N=-mg \tag{2.2}$$

while the moments M_1 and M_2, bending the axis in the planes x_1x_3 and x_2x_3, are

$$M_1 = -A_2(\ddot{\beta} + \ddot{\theta}) - A_1\omega(\dot{\alpha} + \dot{\gamma}),$$
$$M_2 = -A_2(\ddot{\alpha} + \ddot{\gamma}) + A_1\omega(\dot{\beta} + \dot{\theta}) \qquad (2.3)$$

where ω is the angular velocity of proper rotation.

Since the longitudinal force compresses the rotor, the projections u_j (s, t) $(j=1, 2)$ of the deflection of its axis on the coordinate planes x_1x_3 and x_2x_3, must satisfy not (1.7), but rather the differential Eqs.

$$u_j''(s, t) + \lambda_0^2 u_j(s, t) = \frac{1}{EI}[P_j(l-s) + M_j] \quad \text{for } 0 < s < l_1, \ \lambda_0^2 = \frac{mg}{EI}$$

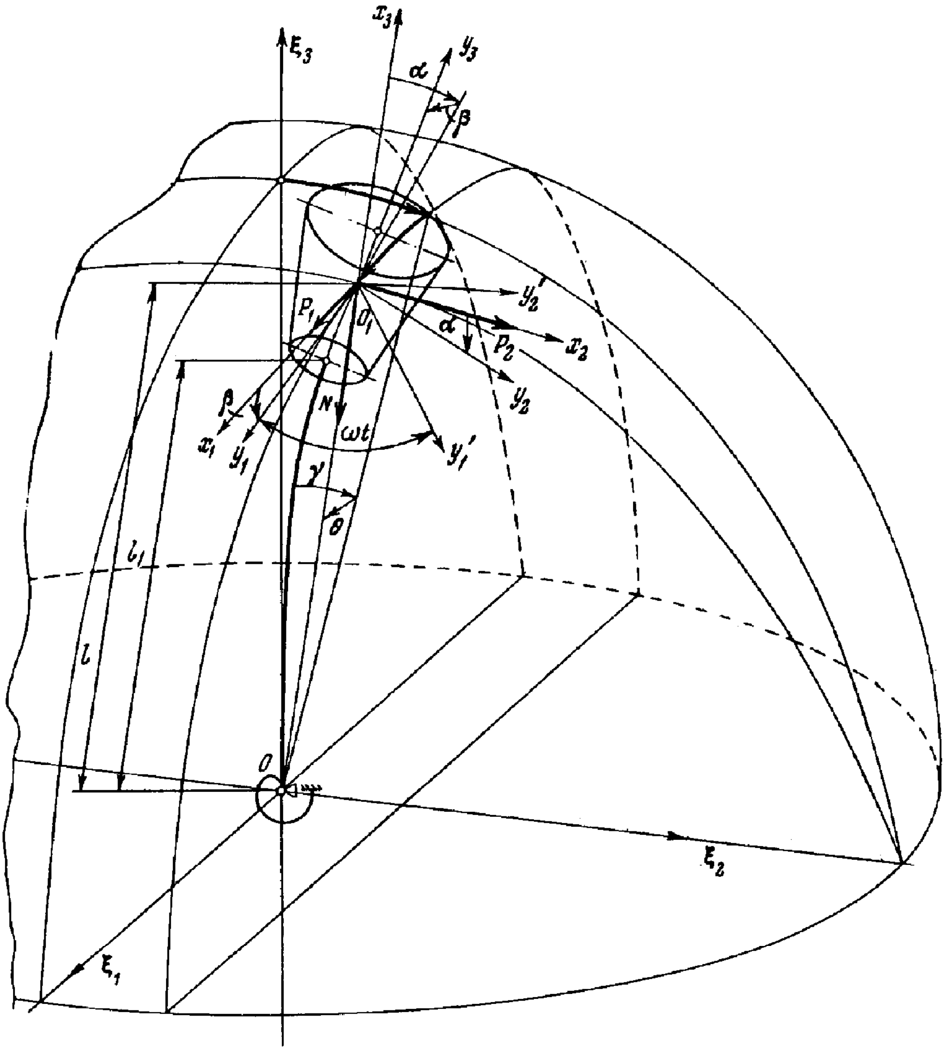

Fig. 5

Integrating these Eqs. under boundary conditions (1.8), we obtain the following expressions for the angles of inclination of the elastic line at the points $s = 0$ and $s = l_1$:

$$mgu_j'(0,\ t) = c_1(\vartheta,\ \vartheta_1)\, P_j + l^{-1} M_j c_2(\vartheta,\ \vartheta_1) \tag{2.4}$$
$$mgu_j'(l,\ t) = c_3(\vartheta,\ \vartheta_1)\, P_j + l^{-1} M_j c_4(\vartheta,\ \vartheta_1)$$

where

$$c_1(\vartheta,\vartheta_1) = (\vartheta/c)\,[\cos\vartheta_1 - (\vartheta - \vartheta_1)\sin\vartheta_1] - 1,\ c_3(\vartheta,\vartheta_1) = \vartheta/c - 1$$
$$c_2(\vartheta,\vartheta_1) = (\vartheta/c)\,[\cos\vartheta_1 - (\vartheta - \vartheta_1)\sin\vartheta_1 - 1],\ c_4(\vartheta,\vartheta_1) = (\vartheta/c)\,(1 - \cos\vartheta_1)$$
$$\vartheta = \lambda_0 l,\ \vartheta_1 = \lambda_0 l_1,\ c = \sin\vartheta_1 + (\vartheta - \vartheta_1)\cos\vartheta_1$$

Let us transform Eqs.(1.6) with the aid of Formulas (2.2) to (2.4) and introduce complex functions (1.13). The equations in variations for steadystate solution (2.1) are

$$c_4\sigma^2 x^{\cdot\cdot} + (c_3 + c_4\sigma^2)\, y^{\cdot\cdot} + gl^{-1}x - gl^{-1}c_3 y - c_4\sigma_0^2\omega i\,(x^{\cdot} + y^{\cdot}) = 0$$
$$(1 - \eta c_2)\sigma^2 x^{\cdot\cdot} + [1 + \sigma^2 - \eta\,(c_1 + \sigma^2 c_2)]\,y^{\cdot\cdot} - \sigma_0^2\omega i\,(1 - \eta c_2)\,(x^{\cdot} +$$
$$+ y^{\cdot}) - gl^{-1}\, y\,[1 - \eta\,(1 + c_1)] = 0 \quad (i = \sqrt{-1}) \tag{2.5}$$

where σ^2, σ_0^2 and η must be determined from (1.14). Making use of substitution (1.16) and introducing dimensionless quantities (1.17), we obtain the frequency Eq.

$$a_0\nu_*^4 + a_1\nu_*^3 + a_2\nu_*^2 + a_3\nu_* + a_4 = 0 \tag{2.6}$$

where the constant coefficients a_k are given by

$$a_0 = [\sin\vartheta_1 - \vartheta_1\cos\vartheta_1 + \eta\vartheta\,(2 - 2\cos\vartheta_1 - \vartheta_1\sin\vartheta_1)]\, c^{-1}\sigma^2$$
$$a_1 = -\omega_* a_0\sigma_0^2/\sigma^2$$
$$a_2 = -1 - c^{-1}\sigma^2\vartheta\cos\vartheta_1 - \eta c^{-1}\,[(1 + \vartheta^2 - \vartheta\vartheta_1 + \sigma^2\vartheta^2)\sin\vartheta_1 - \vartheta_1\cos\vartheta_1]$$
$$a_3 = c^{-1}\sigma_0^2\omega\,(\vartheta\cos\vartheta_1 + \eta\vartheta^2\sin\vartheta_1)$$
$$a_4 = -1 + \eta\vartheta c^{-1}\,[\cos\vartheta_1 - (\vartheta - \vartheta_1)\sin\vartheta_1] \tag{2.7}$$

The necessary condition for stability of the vertical position of the axis of symmetry of the gyroscope is the realness of all four roots of Eq.(2.6). As we know, a polynomial of degree n has n real roots if and only if the coefficients of the leading terms of all $n + 1$ of the Sturm series have the same sign. In the case of a quadrinomial this rule yields the three inequalities

$$3a_1^2 - 8a_0a_2 > 0 \tag{2.8}$$
$$U = a_1^2a_2^2 - 3a_1^3a_3 - 18a_0^2a_3^2 + 14a_0a_1a_2a_3 - 6a_0a_1^2a_4 - 4a_0a_2^3 +$$
$$+ 16a_0^2a_2a_4 > 0$$
$$W = 4\,U^2\,(16\,a_0a_4 - a_1a_3) + 4UV(a_1a_2 - 6\,a_0a_3) - V^2(3a_1^2 - 8a_0a_2) > 0$$

where

$$V = a_1^2a_2a_3 - 4a_0a_2^2a_3 + 3a_0a_1a_3^2 - 9a_1^3a_4 + 32\,a_0a_1a_2a_4 - 48a_0^2a_3a_4$$

In fact, let us apply Euclid's algorithm to the polynomial

$$g(x) = x^4 + ax^3 + bx^2 + cx + d$$

and to its derivative $g'(x)$, each time changing the sign of the remainder. This gives us

$$g(x) = r(x)\,g'(x) - g_1(x), \qquad g'(x) = r_1(x)\,g_1(x) - g_2(x)$$
$$g_1(x) = r_2(x)g_2(x) - g_3$$

We assume that the polynomial $g(x)$ does not have multiple roots, so that g_3 is a constant. All the roots of the polynomial $g(x)$ are real if g_3 and the coefficients of the leading terms of the Sturm functions $g_1(x)$ and $g_2(x)$ are positive.

Dividing $16\,g(x)$ by $g'(x)$, we obtain

$$g_1(x) = (3a^2 - 8b)\,x^2 + 2\,(ab - 6c)\,x + (ac - 16d)$$

Dividing $(3a^2 - 8b)^2\,g'(x)$ by $g_1(x)$ we obtain

$$g_2(x) = 32\,Ax + 16\,B$$

where

$$A = a^2b^2 - 3a^3c + 14\,abc - 6a^2d - 4b^3 - 18c^2 + 16bd$$
$$B = a^2bc - 9a^3d + 3ac^2 + 32abd - 4b^2c - 48cd$$

Finally, we divide $4\,A^2g_1(x)$ by ${}^1/_{16}g_2(x)$, to obtain

$$g_3 = 4\,A^2\,(16d - ac) + 4AB\,(ab - 6c) - B^2\,(3a^2 - 8b)$$

Thus, all the roots of $g(x)$ are real if

$$3a^2 - 8b > 0,\quad A > 0,\quad g_3 > 0 \tag{2.9}$$

In the case under consideration

$$a = a_1/a_0,\ b = a_2/a_0,\ c = a_3/a_0,\ d = a_4/a_0$$

Substituting these values of the coefficients into inequalities (2.9) and multiplying the first of them by a_0^2, the second by a_0^4 and the third by a_0^{10}, we arrive at the three inequalities (2.8), which are the necessary conditions for gyroscope stability.

3. As an example let us consider the stability of a free top with an elastic axis. By a "free" top we mean one which does not have an elastic constraint and the associated restoring moment, i.e. a top for which $\eta = \varkappa = 0$. To simplify analysis we assume that $l \approx l_1$. We then find from (2.7) that

$$a_0 = \sigma^2(1 - f),\quad a_1 = -\omega_*\sigma_0^2(1 - f),\quad a_2 = -(1 + \sigma^2 f),\quad a_3 = \omega_*\sigma_0^2 f,\quad a_4 = -1 \tag{3.1}$$

where the dimensionless parameter f characterizes the relative rigidity of the axis and is

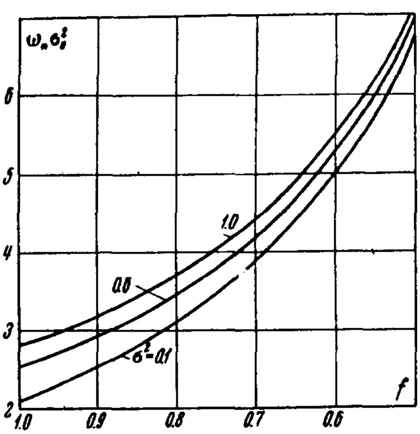

Fig. 6

$$f = \vartheta\,\mathrm{ctg}\,\vartheta,\qquad \vartheta = \lambda_0 l = \left(\frac{mq}{EI}\right)^{1/2} l$$

In the case of an absolutely rigid axis $\vartheta = 0$ and $f = 1$ for small ϑ it is convenient to expand f in powers of ϑ,

$$f = 1 - \sum_{k=1}^{\infty} \frac{2^{2k}\,|\,B_{2k}\,|}{(2k)!}\,\vartheta^{2k}$$

where B_{2k} are Bernoulli numbers,

$$B_0 = 1,\quad B_1 = -{}^1/_2,\quad B_2 = {}^1/_6,\ \ldots$$

In practically important cases the parameter $f > {}^1/_2$, but remains smaller than or equal to unity; it is therefore sufficent to investigate the stability of the top in the range $0 < f \leqslant 1$.

Let us substitute the values of the coefficients from (3.1) into inequalities (2.8). The first of these is satisfied for all angular velocities ω_*. In the case of the first inequality of (2.8) this is self-evident; as regards the second inequality of (2.8), the function U becomes a quadratic polynomial in ω_*^2

$$U(1-f)^{-1} = 3\omega_*^4\sigma_0^8 f(1-f)^2 + \sigma_0^4(1-f)(1+6\sigma^2+10\sigma^2 f-3f^2\sigma^4)\omega_*^2 + \\ +4\sigma^2(1+\sigma^2 f)(1-2\sigma^2 f+\sigma^4 f^2+4\sigma^2) \tag{3.2}$$

which would be negative if $3f^2\sigma^4 > 1+6\sigma^2+10\sigma^2 f$, but, as is easy to show, the discriminant of polynomial (3.2) is positive under this condition. Hence, the condition $U > 0$ is fulfilled for all real values of ω_* in the range of variation of the parameter f under consideration.

Thus, the only necessary condition for the stability of a top with an elastic axis is the inequality $W > 0$. After elementary (though cumbersome) trasformation, the function W turns out to be a fifth-degree polynomial in the parameter $z = \omega_*^2\sigma_0^4$,

$$W(1-f)^{-3} = 36z^5 f^3(1-f)^4 + 3z^4(1-f)^3[-81+108f-f^2(24+36\sigma^2)+160f^3\sigma^2+ \\ +40f^4\sigma^4] + 4z^3(1-f)^2[-(9+648\sigma^2)+f(9+873\sigma^2-216\sigma^4)+f^2\sigma^2(-204+ \\ +9\sigma^2)+f^3\sigma^4(550-9\sigma^2)+308f^4\sigma^6-23f^5\sigma^8]+16z^2\sigma^2(1-f)[-(21+612\sigma^2+ \\ +144\sigma^4)+f(21+828\sigma^2-216\sigma^4)-f^2\sigma^2(200-258\sigma^2+36\sigma^4)+f^3\sigma^4(238-36\sigma^2)+ \\ +f^4\sigma^6(276+123\sigma^2)-59f^5\sigma_8^8-20f^6\sigma^{10}]+256z\sigma^4(1+\sigma^2 f)[-4(1+15\sigma^2+12\sigma^4)+ \\ +4f(1+21\sigma^2+18\sigma^4)-f^2\sigma^2(23+48\sigma^2+12\sigma^4)+f^3\sigma^4(28+20\sigma^2)+ \\ +f^4\sigma^6(-2+12\sigma^2)-8f^5\sigma^8+f^6\sigma^{10}]-1024\sigma^6(1+\sigma^2 f)^2(1+4\sigma^2-2f\sigma^2+f^2\sigma^4)^2$$

With an absolutely rigid axis $f = 1$ and W becomes a linear function of the parameter z

$$W(1-f)^{-3} = 256\sigma^6(1+\sigma^2)^5 z - 1024\sigma^6(1+\sigma^2)^6$$

Moreover, the third inequality of (2.8) becomes $\omega_*^2\sigma_0^4 > 4(1+\sigma^2)$. Hence, recalling (1.14) and (1.17), we obtain the familiar condition for stability of the vertical rotation of an absolutely rigid top $\omega^2 A_1^2 > 4mgl A'_2$, where $A_2' = ml^2 + A_2$ is the equatorial moment of inertia of the gyroscope with respect to the point of support.

Table 1

σ^2	$f=1$	$f=0.9$	$f=0.8$	$f=0.7$	$f=0.6$	$f=0.5$
0.1	4.4	6.406	9.613	15.05	24.99	45.01
0.2	4.8	6.823	10.06	15.54	25.54	45.66
0.4	5.6	7.664	10.96	16.54	26.66	46.97
0.6	6.4	8.514	11.88	17.55	27.80	48.29
0.8	7.2	9.371	12.81	18.56	28.94	49.62
1	8	10.23	13.74	19.59	30.10	50.96
2	12	14.59	18.50	24.85	36.00	57.77
3	16	18.98	23.36	30.25	42.08	64.77

In the range of parameter values under investigation the equation $W=0$ has just the one positive real root z_1. Table 1 contains the values of these roots for various values of the parameters f and σ^2, computed to within four figures.

Fig. 6 shows the dependence of $\sqrt{z_1} = \omega_*\sigma_0^2$ on the parameter f for for three values of σ^2. Below these curves are the zones of unstable vertical rotation of a top with flexible axis ($W < 0$). As we see from the table and Fig. 6, the elastic deformations of the axis enlarge the instability zones considerably. This effort is directly proportional to the flexibility of the axis.

In conclusion we note that the model of a gyroscope considered in the present paper can be used in engineering dynamics, and specifically in investigating the oscillation of vertical rotors in the gravitational field.

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BIBLIOGRAPHY

1. Magnus, K., Die Schwingungen des Kreisels der mit Massen elastisch gekoppelt ist. Proc. of the Conference on Vibration at Göttingen and Kassel, 6-8 October 1938, VDI-Verlag, 1939.
2. Magnus, K., Untersuchungen zur Verminderung störender Rüttelschwingungen an Reiselgeräten. Z. Angew. Math. Mech, Vol. 20, №3, 1940.
3. Maunder. L.. Natural oscillation frequencies of a free gyroscope with an elastic shaft mounted on an elastic suspension. "Mechanics": Collected Translations and Surveys of Foreign Papers, № 5(69), 1961.
4. Whipple, A. P. and Maunder, L., Oscillations of a free gyroscope with an inhomogeneously elastic axis. "Mechanics": Collected Translations and Surveys of Foreign Papers, № 6(88), 1964.
5. Krementulo, V. V., Application of the second method of Liapunov in the investigation of the steady motion of a gyroscope when elastic properties of the rotor axis are taken into account. PMM Vol. 25, № 3, 1961.

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